$\mathcal{IV}$-matching is strongly NP-hard

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Abstract. $\mathcal{IV}$-matching is a generalization of perfect bipartite matching. The complexity of finding $\mathcal{IV}$-matching in a graph was posted as an open problem at the ICALP 2014 conference.

In this note, we resolve the question and prove that, contrary to the expectations of the authors, the given problem is strongly NP-hard (already in the simplest non-trivial case of four layers). Hence it is unlikely that there would be an efficient (polynomial or pseudo-polynomial) algorithm solving the problem.

Keywords: $\mathcal{IV}$-matching, perfect matching, NP-completeness

1 Introduction

The perfect matching problem is one of the central and well studied problems in graph theory. For an exhaustive overview of this area of research we refer to the monograph of Lovász and Plummer [6]. An important part of research is devoted to special classes of graphs (especially to bipartite graphs) as well as various strengthening of the former problem (for example T-joins). For the bipartite setting special attention is focused on min–max type theorems as is for example the pioneering work of König [5].

The complexity status of the $\mathcal{IV}$-MATCHING problem was posted as an open problem in the full version of the article by Fiala, Klavík, Kratochvíl and Nedela [3] at the ICALP 2014 conference. $\mathcal{IV}$-matching is a generalization of perfect bipartite matching to multi-layered graphs. Informally, there is a classical one-to-one matching between layers $2k-1$ and $2k$, but a "two-to-one matching" between layers $2k$ and $2k+1$. There are further restrictions imposed on the matching.

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Motivation. Fiala et al. [3] studied algorithmic aspects of regular graph covers (a graph G covers a graph H if there exists a locally bijective homomorphism from G to H; regular covers are very symmetric covers). The problem of determining for two given graphs G and H whether G regularly covers H generalizes both the graph isomorphism problem and the problem of Cayley graphs [2] recognition. The need to determine whether there is an $\mathcal{IV}$-matching in a given graph appears naturally in an algorithm proposed by the authors. Therefore, they ask whether there is an efficient algorithm for this task.

Our contribution. We show that the problem is strongly NP-complete, already in the simplest non-trivial case of four layers, and therefore a polynomial or pseudo-polynomial algorithm is unlikely. Thus our results imply a dichotomy for the complexity of the $\mathcal{IV}$-MATCHING problem. The concepts around NP are explained for example by Arora and Barak [1].

Outline of the note. A detailed problem definition is given in Section 2 and the proof of NP-completeness is given in Section 3.

2 Problem Definition

The formal definition of $\mathcal{IV}$-matching is as follows.

Definition 1 (Layered graph). *Let $G = (V, E)$ be a bipartite graph. We say that G is a* layered graph *if G fulfills the following conditions:*

- *Vertices are partitioned into ℓ sets $V_1, \dots, V_\ell$ called* layers*. There are only edges between two consecutive layers V_k and V_{k+1} for $k = 1, \dots, \ell - 1$.*
- *Each layer is further partitioned into* clusters*.*
- *Edges of G are described by edges on clusters; we call these edges* macroedges*. If there is a macroedge between two clusters, then vertices of these two clusters induce a complete bipartite graph. If there is no macroedge, then these vertices induce an edge-less graph.*
- *Macroedges between clusters of layers V_{2k} and V_{2k+1} form a matching (not necessarily a maximum matching).*
- *There are no conditions for macroedges between clusters of the layers V_{2k-1} and V_{2k}.*

Definition 2 ($\mathcal{IV}$-matching). *Let G be a layered graph. $\mathcal{IV}$-*matching *is a subset of edges $M \subseteq E$ such that: Each vertex from an even layer V_{2k} is incident to exactly one vertex from $V_{2k-1} \cup V_{2k+1}$. Each vertex from the layer V_{2k-1} is either incident to exactly two vertices from V_{2k-2}, or exactly one vertex from V_{2k} (these two options are exclusive).*

It implies that edges between layers V_{2k-1} and V_{2k} form a matching ($\mathcal{I}$-shapes) and edges between layers V_{2k} and V_{2k+1} form independent $\mathcal{V}$-shapes with centers in the layer V_{2k+1}.

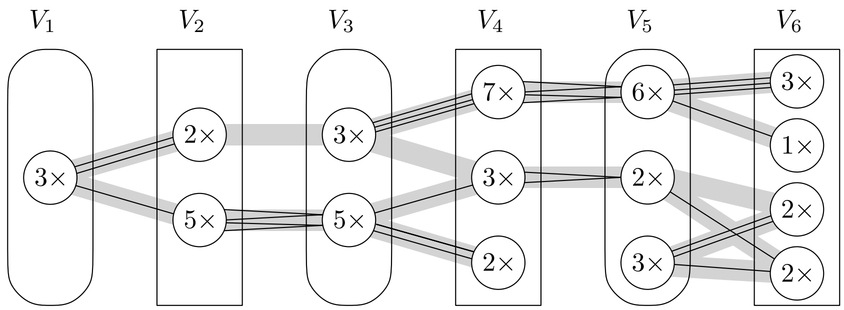

Fig. 1. Layered graph with 6 layers. Numbers denote the number of vertices in a cluster. Macroedges are represented by shaded edges, while edges form an $\mathcal{IV}$-matching.

As the $\mathcal{IV}$-MATCHING problem we denote the decision problem of finding out whether there is an $\mathcal{IV}$-matching in a given graph.

Problem:	$\mathcal{IV}$-MATCHING
Instance:	Graph G described by clusters and macroedges.
Question:	Is there an $\mathcal{IV}$-matching in G?

For $\ell = 2$ the problem is just an ordinary bipartite matching. For odd values of ℓ, the clusters of the last layer V_ℓ can be matched in only one possible way, thus this odd case reduces to the case with $(\ell - 1)$ layers. The first interesting case is therefore $\ell = 4$ and we prove in the next section that this case is already NP-complete.

3 NP-completeness

We prove NP-completeness of the $\mathcal{IV}$-MATCHING problem using a reduction from the 3D-MATCHING problem in this section.

Three-dimensional matching. In the 3D-MATCHING problem we are given a hypergraph $H = (U, F)$. The hypergraph is tripartite, i.e., the set of vertices is partitioned into three equally sized partites X, Y and Z. Each hyperedge consists of exactly one vertex from each partite, thus $F \subseteq X \times Y \times Z$. A set of pairwise disjoint hyperedges covering all vertices is called a *perfect matching*.

Problem:	3D-MATCHING
Instance:	A tripartite hypergraph H.
Question:	Is there a perfect matching in H?

The 3D-MATCHING problem is well-known to be NP-complete; it is actually the seventeenth problem in the Karp's set of 21 NP-complete problems [4].

Theorem 1. *The $\mathcal{IV}$-MATCHING problem is strongly* NP*-complete, already in the case of $\ell = 4$.*

Proof. It is easy to see that the problem is in the class NP: The $\mathcal{IV}$-matching itself is a polynomial certificate and its correctness can be directly verified.

In order to show that the problem is NP-hard, we construct a polynomial-time reduction from the 3D-MATCHING problem to the $\mathcal{IV}$-MATCHING problem. Let the hypergraph $H = (U, F)$ be an instance of the 3D-MATCHING problem with partites X, Y, Z and let us write $n = |X| = |Y| = |Z|$ and $m = |F|$.

We construct the instance $G = (V, E)$ of the $\mathcal{IV}$-MATCHING problem as follows. We put vertices from X and Y into the layer V_1 and vertices from Z into the layer V_4. Each vertex forms its own cluster of the size one. Then for each hyperedge $e = \langle x, y, z\rangle$ we add a cluster with two new vertices x_e, y_e into the layer V_2 and a cluster with one new vertex z_e into the layer V_3. We then add these four edges on clusters: $\langle\{x\}, \{x_e, y_e\}\rangle$, $\langle\{y\}, \{x_e, y_e\}\rangle$, $\langle\{x_e, y_e\}, \{z_e\}\rangle$ and $\langle\{z_e\}, \{z\}\rangle$.

The key idea of our construction is that $\mathcal{V}$s in an $\mathcal{IV}$-matching solution translate to hyperedges *not* present in the perfect matching. An example is given in Fig. 2.

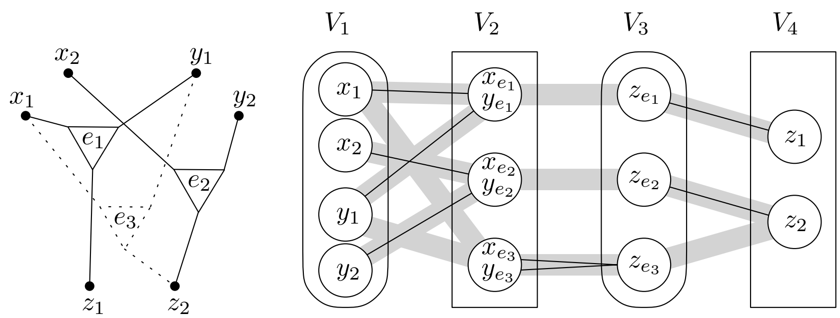

Fig. 2. An instance of the 3D-MATCHING problem with a perfect matching and an equivalent instance of the $\mathcal{IV}$-MATCHING problem instance with the corresponding $\mathcal{IV}$-matching.

The resulting instance of the $\mathcal{IV}$-MATCHING problem has $3n + 3m$ vertices, $4m$ edges and it can be constructed directly in polynomial time. We shall now prove that there is an $\mathcal{IV}$-matching in G if and only if there is a perfect matching in the original hypergraph H.

$\Rightarrow$ Let M be an $\mathcal{IV}$-matching in G. Observe that for each hyperedge $e \in F$ necessarily either all vertices x_e, y_e and z_e are matched with $\mathcal{I}$s or all of them are matched with a single $\mathcal{V}$.

Let us put into our matching of H all hyperedges $e \in F$ such that x_e, y_e and z_e are matched with $\mathcal{I}$s. Because M is an $\mathcal{IV}$-matching, every vertex of the original hypergraph is connected by an $\mathcal{I}$ to exactly one vertex in $V_2 \cup V_3$ and we

chose the corresponding edge to our matching so all vertices of the hypergraph are covered. Because there are $3n$ vertices in $V_1 \cup V_4$ and $3m$ vertices in $V_2 \cup V_3$, the number of $\mathcal{V}$s used is $m - n$ and so we used n hyperedges in our matching of the hypergraph. This proves that we constructed a perfect matching.

$\Leftarrow$ Let $N \subseteq F$ be a perfect matching in the hypergraph H. For each hyperedge $e = \langle x, y, z \rangle$ we connect by $\mathcal{I}$s the pairs of vertices $\{x, x_e\}$, $\{y, y_e\}$ and $\{z, z_e\}$. For each hyperedge $e \notin N$ we cover the vertices x_e, y_e, z_e by a $\mathcal{V}$. Note that every vertex in $V_2 \cup V_3$ is covered by exactly one $\mathcal{I}$ or one $\mathcal{V}$.

Because the matching N covers all vertices in H, every vertex in $V_1 \cup V_4$ is covered by at least one $\mathcal{I}$. Because we put $3n$ $\mathcal{I}$s into the graph, every vertex is covered by exactly one $\mathcal{I}$. This implies that we obtained a correct $\mathcal{IV}$-matching.

This finishes the proof of NP-completeness. Because we only use clusters of size two in the reduction, the problem stays naturally NP-complete when input is encoded in unary. Therefore, strong NP-completeness is established as well. □

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